Robustness of Temporal Logic Specifications for Signals

Georgios Fainekos dissertation series - Part I

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Outline

1. Background and definitions
2. Boolean satisfaction of a specification by a signal - Boolean abstraction
3. Robust satisfaction of a specification by a signal - “Robustness degree”
4. Robust TL semantics - “Robustness estimate”
5. Discrete time signals, timed state sequences and their robustness
6. Continuous time reasoning using discrete time analysis
7. Monitoring algorithm and software tool

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Background

On the use of temporal logic (TL)

- TL useful in software and hardware verification
- But verification undecidable/expensive in continuous and hybrid systems
- Testing of the systems or numerical simulation of the system models are preferred choices in these cases; steady state properties can be fairly easily tested or numerically simulated

Important idea

- Use TL as a specification language for testing
  - Oded Maler and Dejan Nickovic. *Monitoring temporal properties of continuous signals*. FORMATS, 2004
- Advantage: Transient properties can be specified (and hence tested) if we use temporal logic.

Typical structure for testing using TL

Signal

- We either give an analytical formula or some samples of the signal

Specification

- We use metric interval temporal logic to specify some formula
- Observation map: is a Boolean abstraction map from signal space to true/false
- Intuition: We specify that the signal must be within this range during this time span

Monitoring algorithm

- We have some sort of algorithm to check whether or not the signal was indeed within that range during that time span

Result

- If yes, we get a 'true' result; 'false' otherwise

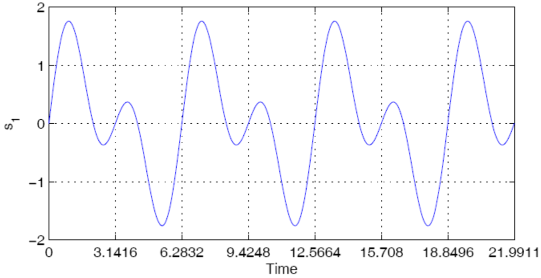
Continuous time signal

Formal definition

- A signal s is a map $s : T \rightarrow X$, where
  - T is a time domain, some subset of $\mathbb{R}_{\geq 0}$
  - X is a metric space, to be defined in the next slide

Example:

$$s_1 = sin(t) + sin(2t), \; T = [0, 7\pi]$$



Metric space, metric, ε-ball

Metric space

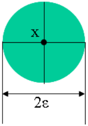
- A metric space (X, d) is an ordered pair of a set X and a metric d.

Metric

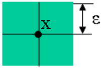
- A metric d is a non-negative function $d : X \times X \to \mathbb{R}_{\geq 0}$, such that $\forall x_1, x_2, x_3 \in X$, we have:
  - $d(x_1, x_2) = 0 \Leftrightarrow x_1 = x_2$
  - $d(x_1, x_2) = d(x_2, x_1)$
  - $d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3)$

ε-ball

- An ε-ball $B_d(x, \varepsilon)$ is defined as
  - $B_d(x, \varepsilon) = \{y \in X | d(x, y) < \varepsilon\}$



Ball in L_2 or Euclidian norm



Ball in L_∞ or *sup* norm

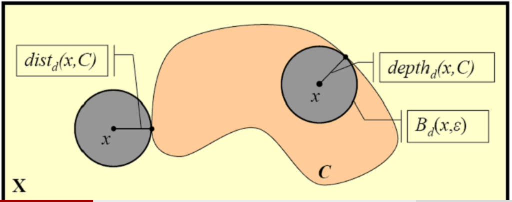
Signed distance

Formal definition

- Let $x \in X$ be a point, $C \subseteq X$ be a set and d be a metric.
- Then, the signed distance from x to C is:
  - $$Dist_d(x, C) = \begin{cases} -dist_d(x, C), & \text{if } x \notin C \\ depth_d(x, C), & \text{if } x \in C \end{cases} \quad (1)$$

    where,
  - $dist_d(x, C) = inf\,\{d(x, y) | y \in Cl(C)\}$
  - $depth_d(x, C) = dist_d(x, X \setminus C)$

Pictorially:



Metric (Interval) Temporal Logic - Syntax

Inductive grammar

$\varphi = \mathsf{T} \mid \perp \mid \mathsf{p} \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \wedge \varphi_2 \mid \varphi_1 \mathcal{U}_{\mathcal{I}} \varphi_2 \mid \varphi_1 \mathcal{R}_{\mathcal{I}} \varphi_2$

Note that:

- In MTL, $\mathcal{I}$ can be any bounded or unbounded but non-empty interval of $\mathbb{R}_{\geq 0}$ e.g. $[a, b], [a, b), (a, b], (a, b)$, where $0 \leq a \leq b$
- In addition, MITL requires $\mathcal{I}$ to be non-singleton, i.e. $a \neq b$
- If $a = 0$ and $b = \infty$, the M(I)TL formula is equivalent to LTL formula
- 'Eventually' and 'Always' operators can be derived as follows:
  - $\Diamond_{\mathcal{I}}\varphi = \mathsf{T}\mathcal{U}_{\mathcal{I}}\varphi$ and $\Box_{\mathcal{I}}\varphi = \perp \mathcal{R}_{\mathcal{I}}\varphi$

Outline

Boolean satisfaction of a specification by a signal

Boolean abstraction

- We specify an observation map
- Observation map labels regions of state space with atomic propositions, e.g. $\mathcal{O}(p_1) = [4, 7]$
  - The signal satisfies p_1 if its value is between 4 and 7, otherwise does not satisfy p_1
- Preimage of observation map: $\mathcal{O}^{-1}(x) = \{p \in AP | x \in \mathcal{O}(p)\}$

Rewriting MITL semantics for testing

- We rewrite $(\mathcal{O}^{-1} \circ s, t) \models \varphi$ as $\ll \varphi, \mathcal{O} \gg = \mathrm{T}$
- If the mapping $\mathcal{O}$ remains constant, we can drop it for brevity and write $\ll \varphi \gg = \mathrm{T}$

Boolean satisfaction of a specification by a signal

Rewriting the MTL grammar for testing:

$$\langle\langle \top \rangle\rangle_C(s,t) := \top$$

$$\langle\langle p \rangle\rangle_C(s,t) := K_{\in}(s(t), \mathcal{O}(p)) = \begin{cases} \top \;\; \text{if } s(t) \in \mathcal{O}(p) \\ \bot \;\; \text{otherwise} \end{cases}$$

$$\langle\langle \neg\phi_1 \rangle\rangle_C(s,t) := \neg\langle\langle \phi_1 \rangle\rangle_C(s,t)$$

$$\langle\langle \phi_1 \vee \phi_2 \rangle\rangle_C(s,t) := \langle\langle \phi_1 \rangle\rangle_C(s,t) \sqcup \langle\langle \phi_2 \rangle\rangle_C(s,t)$$

$$\langle\langle \phi_1 \mathcal{U}_{\mathcal{I}} \phi_2 \rangle\rangle_C(s,t) := \bigsqcup_{t' \in (t +_R \mathcal{I})} \left(\langle\langle \phi_2 \rangle\rangle_C(s,t') \sqcap \prod_{t < t'' < t'} \langle\langle \phi_1 \rangle\rangle_C(s,t'') \right)$$

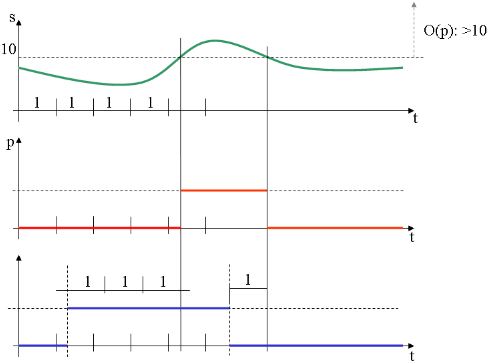
where $t, t', t'' \in R$ *and* $K_{\in}$ *is the characteristic function of the* $\in$ *relation.*

$\sqcap$ means *min* and $\sqcup$ means *max*; Subscript C: continuous time signals

An example

MTL specification $\varphi = \Diamond_{[1,3]} p$

where $\mathcal{O}(p)$ is the set of reals strictly greater than 10



Last graph shows $\ll \varphi \gg (s,t)$ where t is time. When we are talking about $\ll \varphi \gg (s,0)$ we drop 0 for brevity and write $\ll \varphi \gg (s)$

Problems with a Boolean result

Vulnerability to perturbations

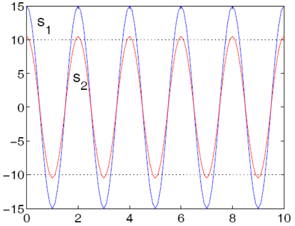


Figure 1.1: Two signals s_1 and s_2 which satisfy the specification: $\Box(\pi_1 \to \Diamond_{\leq 2}\pi_2)$. Here, $\mathcal{O}(\pi_1) = \mathbb{R}_{\leq -10}$ and $\mathcal{O}(\pi_2) = \mathbb{R}_{\geq 10}$.

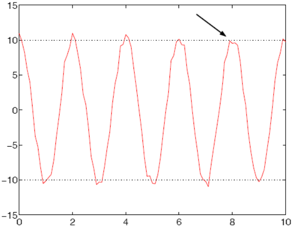


Figure 1.2: The signal s_2 modified by random noise. The arrow points to the point in time where the property fails.

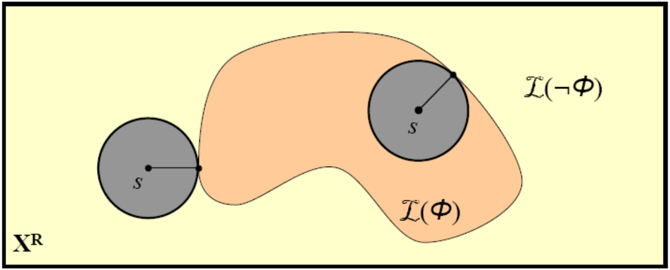
- We cannot distinguish between good and better satisfactions (nor between bad and worse)

Outline

‘Robust’ satisfaction of a specification by a signal

Definition of ‘robustness degree’

- Given a signal s, we define the robustness degree ε as
  - $\varepsilon = Dist_{\rho}(s, \mathcal{L}(\phi))$ [This is a signed distance]
  - where $\rho(s, s') = sup_t \{d(s(t), s'(t)) | t \in T\}$
  - and where $\mathcal{L}(\phi)$ is the set of all signals that satisfy ϕ

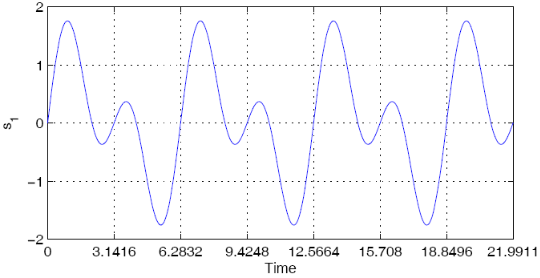


Note that the robustness degree is the radius of the largest (open) ball centered at s that you can fit within $\mathcal{L}(\phi)$

An example where the robustness degree can be computed

A simple example

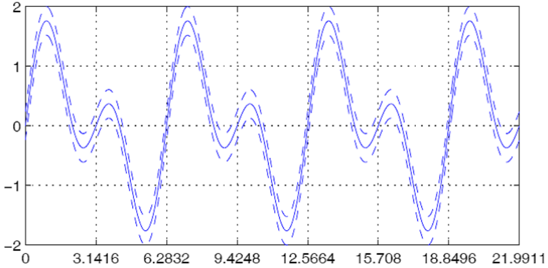
- $s(t) = sin(t) + sin(2t)$
- $\varphi_0 = \Box p_1$ and $\mathcal{O}(p_1) = [-2, 2]$



An example where the robustness degree can be computed

A simple example

- Here, ε can be computed as 0.2398



In general, robustness degree cannot be computed directly, since we don’t know ‘the set of all signals that satisfy the given formula’.

Outline

Multi-valued (aka 'robust') TL semantics

Previous ideas

- De Alfaro et al:
  - Propositions can take values not from $\{0, 1\}$ but in $[0, 1]$
  - Idea used in 'Discounted' model checking - a discount factor between 0 to 1
  - Also used for model checking of say Markov decision processes - transition probabilities between 0 to 1

Idea by Fainekos

- Propositions can take real values
- Details follow...

Robust TL semantics

Inductive grammar

$$[\![\top]\!]_C(s,t) := +\infty$$

$$[\![c]\!]_C(s,t) := c$$

$$[\![p]\!]_C(s,t) := \mathbf{Dist}_d(s(t), \mathcal{O}(p))$$

$$[\![\neg\phi_1]\!]_C(s,t) := -[\![\phi_1]\!]_C(s,t)$$

$$[\![\phi_1 \vee \phi_2]\!]_C(s,t) := [\![\phi_1]\!]_C(s,t) \sqcup [\![\phi_2]\!]_C(s,t)$$

$$[\![\phi_1 \mathcal{U}_{\mathcal{I}} \phi_2]\!]_C(s,t) := \bigsqcup_{t' \in (t +_R \mathcal{I})} \left([\![\phi_2]\!]_C(s,t') \sqcap \bigsqcap_{t<t''<t'} [\![\phi_1]\!]_C(s,t'')\right)$$

$where\ t, t', t'' \in R.$

$\sqcap$ means *inf* and $\sqcup$ means *sup*; Subscript C: continuous time signals

Robustness estimate a lower bound on robustness degree

Important result

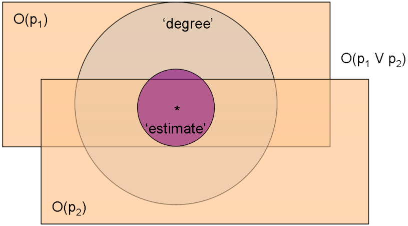
$$\varepsilon_{estimate} = \left| [\![\phi]\!](s) \right| \leq \left| \mathbf{Dist}_{\rho}(s, L(\phi)) \right| = \varepsilon_{actual}$$

which implies

$$\forall s' \in B_\rho(s, \varepsilon), \ll \phi \gg (s', t) = \ll \phi \gg (s, t)$$

‘estimate’ a lower bound on ‘degree’ - why?

By construction of the semantics: An example



- Robustness degree: The radius of the lartest ε-ball we can fit within the $\mathcal{O}(p_1 \vee p_2)$
- Robustness estimate: The semantics ask us to take the *sup* of the radii of the ε-balls we can fit within both observation maps individually.

Outline

Why talk about discrete time signals?

Practical reasons

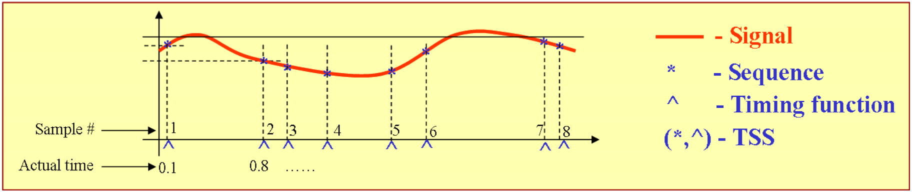
- We may not know the analytical equation of the signal
- We may not have access or to all the (infinite) values of a continuous signal even on a finite real time domain
- All we might have is a number of samples and the corresponding time stamps
- Typical example: result of a variable step numerical ODE integration in Matlab.

Timed State Sequences (TSS)

In words:

- Discrete time signal σ is a sequence of samples, no timing info
- A timing function τ associates time with each sample
- A pair μ=(σ,τ) is called a timed state sequence

Pictorially:



MITL semantics for testing

MITL semantics for TSS

$$\langle\langle \top \rangle\rangle_D(\mu, i) := \top$$

$$\langle\langle p \rangle\rangle_D(\mu, i) := K_\in(\sigma(i), \mathcal{O}(p))$$

$$\langle\langle \neg\phi_1 \rangle\rangle_D(\mu, i) := \neg\langle\langle \phi_1 \rangle\rangle_D(\mu, i)$$

$$\langle\langle \phi_1 \vee \phi_2 \rangle\rangle_D(\mu, i) := \langle\langle \phi_1 \rangle\rangle_D(\mu, i) \sqcup \langle\langle \phi_2 \rangle\rangle_D(\mu, i)$$

$$\langle\langle \phi_1 \mathcal{U}_\mathcal{I} \phi_2 \rangle\rangle_D(\mu, i) := \bigsqcup_{j \in \tau^{-1}(\tau(i)+\mathcal{I})} \left(\langle\langle \phi_2 \rangle\rangle_D(\mu, j) \sqcap \bigsqcap_{i \le k < j} \langle\langle \phi_1 \rangle\rangle_D(\mu, k) \right)$$

where $i, j, k \in N$, $\sigma = \mu^{(1)}$, $\tau = \mu^{(2)}$ *and* $K_\in$ *is the characteristic function of the* $\in$ *relation.*

Robust semantics

Robust semantics for TSS

$$[\![\top]\!]_D(\mu, i) := +\infty$$

$$[\![c]\!]_D(\mu, i) := c$$

$$[\![p]\!]_D(\mu, i) := \mathbf{Dist}_d(\sigma(i), \mathcal{O}(p))$$

$$[\![\neg\phi_1]\!]_D(\mu, i) := -[\![\phi_1]\!]_D(\mu, i)$$

$$[\![\phi_1 \vee \phi_2]\!]_D(\mu, i) := [\![\phi_1]\!]_D(\mu, i) \sqcup [\![\phi_2]\!]_D(\mu, i)$$

$$[\![\phi_1 \mathcal{U}_{\mathcal{I}} \phi_2]\!]_D(\mu, i) := \bigsqcup_{j \in \tau^{-1}(\tau(i)+\mathcal{I})} \left([\![\phi_2]\!]_D(\mu, j) \sqcap \bigsqcap_{i \le k < j} [\![\phi_1]\!]_D(\mu, k)\right)$$

where $i, j, k \in N$, $\sigma = \mu^{(1)}$ *and* $\tau = \mu^{(2)}$.

Outline

Why are DT and CT semantics NOT equivalent?

Consider the DT Until operator

$$\langle\langle \phi_1 \mathcal{U}_{\mathcal{I}} \phi_2 \rangle\rangle_D(\mu, i) := \bigsqcup_{j \in \tau^{-1}(\tau(i)+\mathcal{I})} \left(\langle\langle \phi_2 \rangle\rangle_D(\mu, j) \sqcap \bigsqcap_{i \leq k < j} \langle\langle \phi_1 \rangle\rangle_D(\mu, k) \right)$$

where $i, j, k \in N$, $\sigma = \mu^{(1)}$ *and* $\tau = \mu^{(2)}$.

Observations:

- The actual interval and the samples within that interval do not coincide
- If we have no sample within some interval, TSS cannot capture the properties of the original signal

Can we force DT and CT equivalence?

Strengthening of formulas

- Idea introduced by Huang et al
  - Jinfeng Huang, Jeroen Voeten, and Marc Geilen, *Real-time property preservation in approximations of timed systems*, Conference on Formal Methods and Models for Co-Design, 2003
- Satisfaction of a strengthened formula by a TSS will guarante (under certain assumptions) the satisfaction of the original formula by the signal

Assumptions on signal behavior - bounded spread

- Intuitively: Signal doesn't spread infinitely in a finite duration

Assumptions for sampling - at least one sample per interval

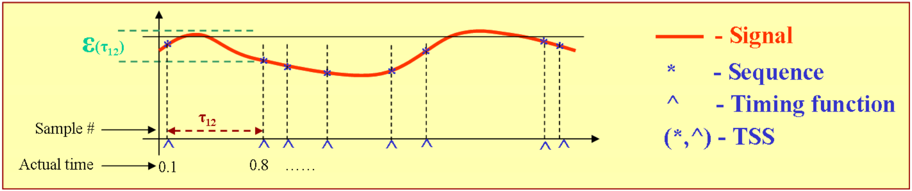
- We have enough data to build on

Let us look at these assumptions in detail...

Well-behavedness assumptions

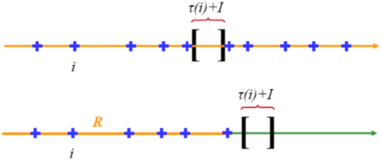
Bounded spread

$\forall t, t' \in R, d(s(t), s(t')) \leq \mathcal{E}(|t - t'|)$, R: signal domain on real number line



At least one sample per interval

Disallowed cases (pictorially):



No sample in the interval

Empty intersection with the signal domain R

Strengthening of formulas with resepect to time

Main idea: Satisfying strenghened formula in DT $\Rightarrow$ satisfying the original (weaker) formula in CT

Atomic predicates and their Boolean combinations

No direct strenghening needed (no 'interval' for these operators)

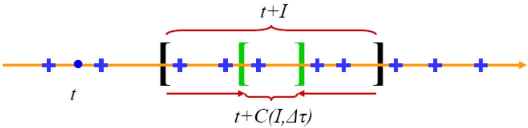
- $\mathbf{str}_{\Delta\tau}(p) = p$
- $\mathbf{str}_{\Delta\tau}(\neg p) = \neg p$
- $\mathbf{str}_{\Delta\tau}(\varphi_1 \vee \varphi_2) = \mathbf{str}_{\Delta\tau}(\varphi_1) \vee \mathbf{str}_{\Delta\tau}(\varphi_2)$
- $\mathbf{str}_{\Delta\tau}(\varphi_1 \wedge \varphi_2) = \mathbf{str}_{\Delta\tau}(\varphi_1) \wedge \mathbf{str}_{\Delta\tau}(\varphi_2)$

Let's see the cases where strengthening is needed, in detail on the next slide...

Strengthening of formulas with respect to time

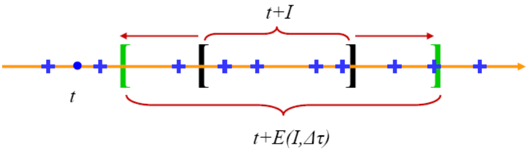
'Until' operator: Compress by $\Delta\tau$ [where $\Delta\tau = sup_i(\tau_{i+1} - \tau_i)$]

- $\mathbf{str}_{\Delta\tau}(\varphi_1 \mathcal{U}_{\mathcal{I}} \varphi_2) = \mathbf{str}_{\Delta\tau}(\varphi_1)\ \mathcal{U}_{C(\mathcal{I},\Delta\tau)}\ \mathbf{str}_{\Delta\tau}(\varphi_2)$



'Release' operator: Expand by $\Delta\tau$

- $\mathbf{str}_{\Delta\tau}(\varphi_1 \mathcal{R}_{\mathcal{I}} \varphi_2) = \mathbf{str}_{\Delta\tau}(\varphi_1)\ \mathcal{R}_{E(\mathcal{I},\Delta\tau)}\ \mathbf{str}_{\Delta\tau}(\varphi_2)$



DT - CT equivalence

Given a specification and a TSS, if

- we know the value of $\Delta\tau$ and strengthen the specification by $\Delta\tau$ and
- well-behavedness assumptions are satisfied post-strengthening and
- we (somehow) know $\mathcal{E}(\Delta\tau)$ and
- and we find the robustness estimate of the given TSS on this strengthened specification

and if

- the robustness estimate of the TSS for the strengthened specification turnes out to be greater than $\mathcal{E}(\Delta\tau)$

then

- the original continuous time signal satisfies the original specification

Outline

Monitoring algorithm

A recursive algorithm

Algorithm 1 Monitoring the Robustness of Timed State Sequences

Input: An MTL formula ϕ, a finite timed state sequence $\mu = (\sigma, \tau)$ and a predicate map $\mathcal{O}$

Output: The formula's robustness estimate

procedure MONITOR($\phi, \mu, \mathcal{O}$)
    $i \leftarrow 0$
    while $\phi \neq \varepsilon \in \overline{\mathbb{R}}$ **do** $\qquad \triangleright$ ϕ has not been reduced to a value
        if $i < \max \mathbf{dom}(\tau)$ **then** $\phi \leftarrow$ DERIVE($\phi, \sigma(i), \delta\tau(i), \perp, \mathcal{O}$)
        else $\phi \leftarrow$ DERIVE($\phi, \sigma(i), 0, \top, \mathcal{O}$)
        end if
        $i \leftarrow i + 1$
    end while
end procedure

Monitoring algorithm

A recursive algorithm (continued...)

Algorithm 2 Deriving the Future

Input: The MTL formula ϕ, the current value of the signal x, the time period δt before the next value in the signal, a variable *last* indicating whether the next state is the last and the predicate map $\mathcal{O}$

Output: The MTL formula ϕ that has to hold at the next moment in time

procedure $\textsc{Derive}(\phi, x, \delta t, last, \mathcal{O})$
  if $\phi = \top$ **then return** $+\infty$
  else if $\phi = \varepsilon \in \overline{\mathbb{R}}$ **then return** ε
  else if $\phi = p \in AP$ **then return** $\mathbf{Dist}_d(x, \mathcal{O}(p))$
  else if $\phi = \neg\phi_1$ **then return** $\neg\textsc{Derive}(\phi_1, x, \delta t, last, \mathcal{O})$
  else if $\phi = \phi_1 \vee \phi_2$ **then**
    return $\textsc{Derive}(\phi_1, x, \delta t, last, \mathcal{O}) \vee \textsc{Derive}(\phi_2, x, \delta t, last, \mathcal{O})$
  else if $\phi = \phi_1 \mathcal{U}_{\mathcal{I}} \phi_2$ **then**
    $\alpha \leftarrow K_{\in}^{\infty}(0, \mathcal{I}) \wedge \textsc{Derive}(\phi_2, x, \delta t, last, \mathcal{O})$
    if $last = \top$ **then return** α
    else return $\alpha \vee (\textsc{Derive}(\phi_1, x, \delta t, last, \mathcal{O}) \wedge \phi_1 \mathcal{U}_{(-\delta t)+_R \mathcal{I}} \phi_2)$
    end if
  end if
end procedure

Software tool

TaLiRo

- Computes the robustness estimate
- Takes MTL specifications as input
- Can handle 1D signals as of now
- Can handle polytopic observation maps
- Available at:

  `http://www.seas.upenn.edu/~fainekos/robustness.html`

Summary

Take-away messages from this talk

- Multi-valued TL semantics make the use of TL more robust in testing
- Hopefully could help to popularize the use of TL beyond purely discrete systems, into continuous and hybrid systems :-)

Stay tuned for part II talk

- Exciting new extensions possible
- We will discuss: Verification using robust testing
- We will also briefly review approximate bisimulations
- Stay tuned. . .

References

Georgios Fainekos's work

- All credit should go to Georgios Fainekos, this is his work.
- On the other hand, if there were any mistakes, they were most likely mine.

References

- Some figures and formulas were taken from the thesis and talk slides by Georgios.
- The references i.e. the presentations, publications and other interesting reference material is available at:

  `http://www.seas.upenn.edu/~fainekos/the_public.html`

Thank you

Thank you

- Thanks to Ed Clarke for hosting me
- Thanks to Bruce Krogh and Alex Donzé for reviewing the slides
- Thank you all for attending